

Date 2013 12 31

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \dots \mid a_i \in \{0, 1, \dots, p-1\}\}$$

$$\mathbb{Q}_p = \{a_{-k} p^{-k} + a_{-k+1} p^{-(k-1)} + \dots + a_1 p^{-1} + a_0 + \dots \mid a_i \in \{0, 1, \dots, p-1\}\}$$

$$r \quad (a_{-k} \neq 0)$$

$$|v|_p = \left(\frac{1}{p}\right)^{-k}$$

$$\bar{\mathbb{Q}}_p = \mathbb{Q}_p (\alpha / \alpha, \text{roots of } f(x) \in \mathbb{Q}_p[x])$$

\mathbb{Q}_p = p -adic completion of $\bar{\mathbb{Q}}_p$ ← this is algebraically closed

Theorem: The non-equivalent absolute values of \mathbb{Q} are $\{|l_\infty|\}, |l_2|, |l_p|, \dots, |l_p|$

$$\text{Eg. } n \in \mathbb{Z}_{>0} \quad n = p_1^{a_1} \cdots p_k^{a_k} \quad \prod_{p \in \{2, 3, \dots, \infty\}} |n|_p = 1$$

8. Zeta function of affine toric hypersurfaces

$q = p^r$ prime power, \mathbb{F}_q finite field of q elements

$\Delta \subseteq \mathbb{R}^n$ n -dim integral polytope (This is in \mathbb{Z}^n)

$$f \in \mathbb{F}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$f = \sum_{u \in \Delta \cap \mathbb{Z}^n} a_u x^u, x^u = x_1^{u_1} \cdots x_n^{u_n}, u_i \in \mathbb{Z}, a_u \in \mathbb{F}_q$$

$\Delta(f) = \Delta$ $\Delta(f) =$ the convex closure in \mathbb{R}^n of the exponents u with $a_u \neq 0$
i.e. $u \in \Delta$ if u is a vertex of Δ .

$$U_f := \{f(x_1, \dots, x_n) = 0\} \hookrightarrow \mathbb{G}_m^n$$

$$\# U_f(\mathbb{F}_{q^k}) = \#\{(x_1, \dots, x_n) \in (\mathbb{F}_{q^k})^n \mid f(x_1, \dots, x_n) = 0\} \quad (k=1, 2, \dots)$$

$$Z(U_f, T) = \exp \left(\sum_{k=1}^{\infty} \frac{T^k}{k} \# U_f(\mathbb{F}_{q^k}) \right) \in \mathbb{Q}(T) \quad (\text{Dirichlet})$$

Goal: a precise description of $Z(U_f, T)$

Def: ① If $\Delta' \subset \Delta$ is a face of Δ , define $f^{\Delta'} = \sum_{u \in \Delta' \cap \mathbb{Z}^n} a_u x^u$

f is called Δ -regular if for every face Δ' (of any dim) of Δ the system $f^{\Delta'} = x_1 \frac{\partial f^\Delta}{\partial x_1} = \dots = x_n \frac{\partial f^\Delta}{\partial x_n} = 0$ has no solution in $(\mathbb{F}_q^\times)^n$

Thm (GKZ) ① \exists non-zero polydiscs $\in \mathcal{Z}[t_u \mid u \in \Delta \cap \mathbb{Z}^n]$

s.t. f is Δ -regular over $\mathbb{F}_q \iff$

$$\text{disc}_\Delta(f) = \text{disc}_\Delta(a_u) \neq 0 \text{ in } \mathbb{F}_q$$

② $\Delta(\text{disc}_\Delta)$ (secondary polytope of Δ) is determined explicitly

Question: For which prime p , ~~such that~~ $\text{disc}_\Delta \equiv 0 \pmod{p}$

Def: Let $C(\Delta)$ be the cone in \mathbb{R}^{n+1} generated by 0 and $(1, \Delta)$

$$\text{i.e. } C(\Delta) = \bigcup_{c \geq 0} c(1, \Delta)$$

① For $k = 0, 1, 2, \dots$ $W_\Delta(k) = \#\{(k, k\Delta) \cap \mathbb{Z}^{n+1}\}$

= # of lattice points on $(k, k\Delta)$

= dimension of the hom. deg ~~of~~ k part
of the graded algebra $A_\Delta = \sum_{k \geq 0} \sum_{u \in k\Delta} \mathbb{C} x^u$

② The Hodge numbers of Δ are

$$h_\Delta(k) = W_\Delta(k) - \binom{n+1}{1} W_\Delta(k-1) + \binom{n+2}{2} W_\Delta(k-2) \dots$$

$$h_\Delta(k) = 0 \text{ if } k \geq n+1$$

$h_\Delta(k) = \dim_{\mathbb{C}}$ of the hom. deg k part of

the quadratic \mathbb{C} -algebra $A_\Delta / (f, x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n})$

(f is Δ -regular)

$$\textcircled{3} \deg(\Delta) = d(\Delta) := n! \text{ Vol}(\Delta) = \sum_{k=0}^n h_\Delta(k)$$

Thm (Adolphson - Speiser, 1989 Denef - Loeser, 1992)

Assume f is Δ -regular over $\mathbb{F}_q \Rightarrow$

$$\textcircled{1} \#(U_f, T) = \prod_{i=0}^{n-1} (1 - q^i T)^{\binom{-1}{i} \binom{n}{i}} \cdot P_f(T)^{\binom{1-i}{n}}$$

where $P_f(T) \in 1 + T \mathbb{Z}[T]$ of degree $d(\Delta) - 1$

$$\textcircled{2} \text{ write } P_f(T) = \prod_{i=1}^{d(\Delta)-1} (1 - \alpha_i(f) T), \alpha_i(f) \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

$$\Rightarrow |\alpha_i(f)| \leq \sqrt{q}^{n-1}$$

The precise weights of $\alpha_i(f)$ were also determined by Denef - Loeser

Let ℓ be a prime, embed $\alpha_i \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$

If $\ell \neq p = \text{char}(\mathbb{F}_q) \Rightarrow |\alpha_i(f)|_\ell = 1 \quad \text{ord}_\ell \alpha_i(f) = 0$

If $\ell = p \Rightarrow |\alpha_i(f)|_p = ? \quad \text{ord}_p \alpha_i(f) = ? \quad (q\text{-slope})$

Corollary: f is Δ -regular / $\mathbb{F}_q \Rightarrow$

$$\left| \#(U_f, \mathbb{F}_{q^k}) - \frac{(q^k - 1)^{n-1} (-1)^k}{q^k} \right| \leq (d(\Delta) - 1) \sqrt{q}^{k(n-1)}$$

9. Newton polygon and Hodge polygon

write $P_f(T) = 1 + c_1 T + \dots + c_{d(\Delta)-1} T^{d(\Delta)-1}, c_i \in \mathbb{Z}$

Def: The q -adic Newton polygon of $P_f(T)$ is the lower convex closure in \mathbb{R}^2 of the points $(k, \text{ord}_q(c_k)), k=0, \dots, d(\Delta)-1$ denoted by $\underline{NP}(f) (= NP(f \otimes \mathbb{F}_{q^m}))$
Newton polygon

Prop: Let h_s = the horizontal length of the slope s side

$\Rightarrow P_f(T)$ has precisely h_s reciprocal roots $\alpha_i(f)$ s.t.

$$\text{ord}_T(d\alpha_i(f)) = s \quad \forall s \in \mathbb{Q}_{\geq 0}$$

Question: Determine $NP(f)$ (f is Δ -regular)

Def. The Hodge polygon of Δ , denoted by $HP(\Delta)$ is

the polygon in \mathbb{R}^2 with a side of slope $k-1$

with horizontal length $h_\Delta(k)$ $1 \leq k \leq n$

The vertices are $(0,0), (\sum_{m=1}^k h_\Delta(m), \sum_{m=1}^k (m-1)h_\Delta(m))$ $k=1, \dots, n$

Thm (AS): $NP(f) \geq HP(\Delta)$

Def: If $NP(f) = HP(\Delta) \Rightarrow f$ is called ordinary

Question: when f is ordinary i.e. when $NP(f) = HP(\Delta)$?

Def: $M_p(\Delta) = \{f \in \widehat{\mathbb{F}_p}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \mid \Delta(f) = \Delta, f \text{ is } \Delta\text{-regular over } \widehat{\mathbb{F}_p}\}$

Thm (Grothendieck) \exists a generic Newton polygon $GNP(\Delta, p)$
s.t. $GNP(\Delta, p) = \inf \{NP(f) \mid f \in M_p(\Delta)\}$

Cor For $f \in M_p(\Delta) \Rightarrow NP(f) \geq GNP(\Delta, p) \geq HP(\Delta)$

Question: Given Δ . for which p , $GNP(\Delta, p) = HP(\Delta)$

Conj: (AS, 1989) For $p \gg 0$, $GNP(\Delta, p) = HP(\Delta)$

Thm (W, 1993) The AS Conj is true if $n \leq 3$, false if $n \geq 4$

E.g. $f = a_0 + a_1 x_1 + \dots + a_n x_n + a_{n+1} x_1 \cdots x_n \quad a_i \neq 0$

$$\Delta = \text{Conv}\{(0, \dots, 0), (1, 0, \dots, 0), \dots, (0, \dots, 1), (1, \dots, 1)\}$$

$\Rightarrow d(\Delta) = n$ ($n \geq 2$) Δ is an empty simplex (no lattice points other than vertex)

~~GNP(f) = P~~

$$NP(f) = GNP(\Delta, p) (p + \deg) \Rightarrow \text{AS is false if } n \geq 4$$

$$= HP(\Delta) \Leftrightarrow p \equiv 1 \pmod{n-1}$$

10. Decomposition thm.

Def. ① A triangulation of Δ is a decomposition

$$\Delta = \bigcup_{i=1}^n \Delta_i$$

s.t. each Δ_i is a simplex, $\Delta_i \cap \Delta_j$ is a common face
for both Δ_i, Δ_j

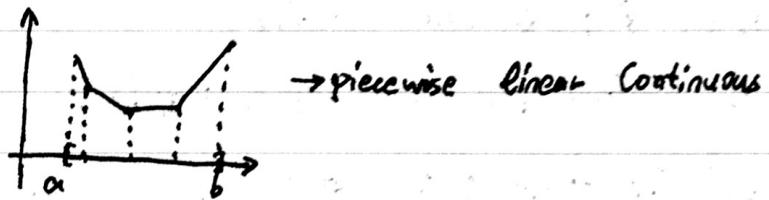
② The triangulation in ① is convex (coherent)

if \exists piecewise linear function $\phi: \Delta \rightarrow \mathbb{R}$

s.t. ① ϕ is convex i.e. $\phi\left(\frac{1}{2}x + \frac{1}{2}x'\right) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x') \quad \forall x, x' \in \Delta$

② The domain of linearity of ϕ are
precisely the n -dim simplex Δ_i ($1 \leq i \leq m$)

Eg:



Thm. ① Let $\Delta = \bigcup_{i=1}^m \Delta_i$ be a convex integral triangulation

of Δ

If $GNP(\Delta_i, p) = HP(\Delta_i) \quad \forall 1 \leq i \leq m$

$\Rightarrow GNP(\Delta, p) = HP(\Delta)$

② If Δ is a simplex, and $p \equiv 1 \pmod{d(\Delta)}$

$\Rightarrow GNP(\Delta, p) = HP(\Delta)$

Cor.: let $\Delta = \bigcup_{i=1}^m \Delta_i$ be a convex integral triangulation

If $p \equiv 1 \pmod{\text{lcm}(\frac{d(\Delta_1)}{d(\Delta)}, \dots, \frac{d(\Delta_m)}{d(\Delta)})}$ $\Rightarrow GNP(\Delta, p) = HP(\Delta)$

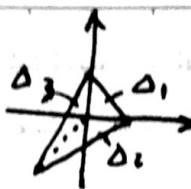
Eg. (Dwork family of CY hypersurface)

$$f(x_1, \dots, x_n) = x_1 + x_2 + \dots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda$$

$$\Rightarrow GNP(\Delta, p) = HP(\Delta) \quad \forall p$$

$n=2$

$$d(\Delta_i) = 1$$



$$\Rightarrow P \equiv 1 \pmod{1} \Rightarrow GNP(\Delta, P) = HP(\Delta)$$

$\text{Ex } \Delta = \{(d, 0, \dots, 0), (0, d, \dots, 0), (0, \dots, 0, d), (0, \dots, 0)\}$

$$\Rightarrow GNP(\Delta, P) = HP(\Delta) \quad \forall P, n$$

\Rightarrow The universal family of affine (projective) hypersurfaces of degree d in A^n is generated generally ordinary $U_{P,d,n}$

Cor ① If $n = \dim(\Delta) \geq 2 \Rightarrow GNP(\Delta, P) = HP(\Delta)$

② If $n = \dim(\Delta) = 3 \Rightarrow GNP(\Delta, P) = HP(\Delta)$ if $P > 6 \text{Vol}(\Delta)$

Def let Δ be n -dim integral convex in R^n

Assume $0 \in \text{Interior}(\Delta)$. The dual of Δ is

$$\Delta^* = \{(x_1, \dots, x_n) \in R^n \mid \sum x_i y_i \geq 1 \quad \forall (y_1, \dots, y_n) \in \Delta\}$$

Δ^* is also convex polytope, not necessarily integral

$$(\Delta^*)^* = \Delta$$

Def n -dim integral convex $\Delta \subseteq R^n$ s.t. $0 \in \text{Int}(\Delta)$

is called reflexive if Δ^* is also integral

(\Rightarrow Δ defines a toric CY hypersurface)

Thm If Δ is reflexive and $\dim(\Delta) \leq 4$

$$\Rightarrow GNP(\Delta, P) = HP(\Delta) \quad \forall P \gg 0$$

This result should have application in arithmetic mirror symmetry

11. Slope zeta function and mirror symmetry
 Let (X, Y) be a mirror pair of CY/\mathbb{F}_q

One hopes for 3-dim CY mirror pair $/\mathbb{F}_{q^k}$

$$Z(X, T) = Z(Y, T)^{-1} \Rightarrow \text{not true}$$

Question: How to modify $Z(X, T)$ s.t. the desired relation is true

$$\text{Eg: } X_\lambda: \underbrace{x_0^{m+1} + x_1^{m+1} + \dots + x_n^{m+1}}_{Y_\lambda} + \lambda x_0 x_1 \dots x_n = 0 \quad \lambda \in \mathbb{F}_q$$

$$Y_\lambda: \left\{ x_1 + x_2 + \dots + x_n + \frac{\lambda}{x_1 \dots x_n} + \lambda = 0 \right\}$$

$$\Rightarrow \# X_\lambda(\mathbb{F}_{q^k}) \equiv \# Y_\lambda(\mathbb{F}_{q^k}) \pmod{q^k}$$

Def For X/\mathbb{F}_q , $Z(X, T) = \prod_i (1 - \alpha_i T)^{\pm 1} \quad \alpha_i \in \mathbb{C}_p$

① The slope zeta function of X/\mathbb{F}_q is

$$S(X, U, T) = \prod_i (1 - U^{\text{ord}_q(\alpha_i)} T)^{\pm 1}$$

one hopes $S(X, U, T) \stackrel{?}{=} \frac{1}{S(Y, U, T)}$ if $n=3$

② If $f: X \rightarrow S/\mathbb{F}_q$ (a family)

the slope-zeta function of f is the generic one among $S(f^{-1}(s), U, T)$ $\forall s \in S$

Conj: Let X be a 3-dim CY over \mathbb{Q}

Assume X has a mirror pair over \mathbb{Q}

\Rightarrow the universal family containing X is generically ordinary $\forall p > 0$

\Rightarrow Conj: (arithmetic mirror conj)

Let $\{f, g\}$ be two generic mirror family of CY over \mathbb{Q}

$$\text{Then } \forall p > 0, \quad S(f \otimes \mathbb{F}_p, U, T) = \frac{1}{S(g \otimes \mathbb{F}_p, U, T)}$$